

THE REALIZATION OF THE DEGREE ZERO PART OF THE MOTIVIC POLYLOGARITHM ON ABELIAN SCHEMES IN DELIGNE-BEILINSON COHOMOLOGY

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ABSTRACT. We use Burgos' theory of arithmetic Chow groups to exhibit a realization of the degree zero part of the polylogarithm on abelian schemes in Deligne-Beilinson cohomology.

1. INTRODUCTION

1.1. The degree zero part of the motivic polylogarithm. In [KR14], G. Kings and D. Rössler have given a simple axiomatic description of the degree zero part of the polylogarithm on abelian schemes. We briefly recall it here.

In [Sou85], C. Soulé has defined motivic cohomology for any variety V over a field

$$H_{\mathcal{M}}^i(V, j) := \mathrm{Gr}_{\gamma}^j K_{2j-i}(V) \otimes \mathbb{Q}.$$

Now let $\pi : \mathcal{A} \rightarrow S$ be an abelian scheme of relative dimension g , let $\varepsilon : S \rightarrow \mathcal{A}$ be the zero section, $N > 1$ an integer and let $\mathcal{A}[N]$ be the finite group scheme of N -torsion points. Here S is smooth over a subfield k of the complex numbers. For any integer $a > 1$ and any $W \subseteq \mathcal{A}$ open sub-scheme such that

$$j : [a]^{-1}(W) \hookrightarrow W$$

is an open immersion (here $[a] : \mathcal{A} \rightarrow \mathcal{A}$ is the a -multiplication on \mathcal{A}), the trace map with respect to a is defined as

$$(1) \quad \mathrm{tr}_{[a]} : H_{\mathcal{M}}(W, *) \xrightarrow{j^*} H_{\mathcal{M}}([a]^{-1}(W), *) \xrightarrow{[a]_*} H_{\mathcal{M}}(W, *)$$

For any integer r we let

$$H_{\mathcal{M}}(W, *)^{(r)} := \{\psi \in H_{\mathcal{M}}(W, *) \mid (\mathrm{tr}_{[a]} - a^r \mathrm{Id})^k \psi = 0 \text{ for some } k \geq 1\}$$

be the generalized eigenspace of $\mathrm{tr}_{[a]}$ of weight r .

Then the zero step of the motivic polylogarithm is a class in motivic cohomology

$$\mathrm{pol}^0 \in H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g).$$

To describe it more precisely, consider the residue map along $\mathcal{A}[N]$

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \varepsilon(S), 0).$$

This map induces an isomorphism

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \cong H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \varepsilon(S), 0)^{(0)}$$

(see Corollary 2.2.2 in [KR14]) and pol^0 is the unique element mapping to the fundamental class of $\mathcal{A}[N] \setminus \varepsilon(S)$.

Now let us consider the map $\mathrm{cyc}_{\mathrm{an}}$ defined as the composition

(2)

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \xrightarrow{\mathrm{cyc}} H_D^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{\mathrm{forgetful}} H_{D,\mathrm{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$$

where cyc is the regulator map into Deligne-Beilinson cohomology and the second map is the forgetful map from Deligne-Beilinson cohomology to analytic real Deligne cohomology (see the end of section 2 for the notations used here).

In [MR11], V. Maillot and D. Rössler constructed a canonical class of currents $\mathfrak{g}_{\mathcal{A}^\vee}$ on \mathcal{A} (cf. Theorem 5 in Section 5) which gives rise to a class in analytic Deligne cohomology

$$([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]} \in H_{D,\mathrm{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)).$$

This element is represented by $\frac{\gamma}{(2\pi i)^{1-g}}$, where γ is any smooth form on $\mathcal{A} \setminus \mathcal{A}[N]$ in the class $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$.

The main result in [KR14] is the following.

Theorem 1. *We have*

$$-2 \cdot \mathrm{cyc}_{\mathrm{an}}(\mathrm{pol}^0) = ([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}.$$

Furthermore the map

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \rightarrow H_{D,\mathrm{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$$

induced by $\mathrm{cyc}_{\mathrm{an}}$ is injective.

1.2. Our main result. In this paper we give a refinement of Theorem 1 supposing S is proper over k (see Corollary 8 and Theorem 10).

Before stating our result, we recall that in [Bur97], Burgos introduced a complex that naturally computes the Deligne-Beilinson cohomology: this is the complex $E_{\log}^*(\cdot)$ of smooth forms with logarithmic singularities along infinity (see section 3.1 for its definition).

Theorem 2. *Let S be proper over k . The class of currents $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ has a representative which is smooth on $\mathcal{A} \setminus \mathcal{A}[N]$ and has logarithmic singularities along infinity. Any such η defines an element*

$$\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_D^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \right)$$

which does not depend on the choice of η . This element verifies

$$-2 \cdot \text{cyc}(\text{pol}^0) = \frac{\tilde{\eta}}{(2\pi i)^{1-g}}$$

and

$$\text{forgetful} \left(\frac{\tilde{\eta}}{(2\pi i)^{1-g}} \right) = ([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}.$$

1.3. An outline of the paper. Let us now give an outline of the contents of each section.

In section 2 we review some notations and definitions coming from Arakelov theory.

In section 3 we recall Burgos' theory of arithmetic Chow groups.

Sections 4, 5 and 6 contain the proof of Theorem 2. This proof combines two arguments. On one hand we use Burgos' theory in order to prove an interesting intermediate result which relates the classical arithmetic Chow groups to Deligne-Beilinson cohomology (see Theorem 3 in section 4). On the other hand we do some calculations using the current $\mathfrak{g}_{\mathcal{A}^\vee}$ in order to prove that the class of

$$T := [N]^*(\varepsilon(S), \mathfrak{g}_{\mathcal{A}^\vee}) - N^{2g}(\varepsilon(S), \mathfrak{g}_{\mathcal{A}^\vee})$$

in $\widehat{\text{CH}}^g(\mathcal{A})_{\mathbb{Q}}$ is zero (cf. Proposition 6 in section 5). This proposition will allow us to apply Theorem 3 to our case (see section 6).

2. NOTATIONS

We begin with a review of some notations and definitions coming from Arakelov theory (see Sections 1,2,3 in [GS90] for a compendium).

Let (R, Σ, F_∞) be an arithmetic ring i.e.

- R is an excellent regular Noetherian integral domain,
- Σ is a finite nonempty set of monomorphisms $\sigma : R \rightarrow \mathbb{C}$,
- F_∞ is an anti-linear involution of the \mathbb{C} -algebra $\mathbb{C}^\Sigma := \underbrace{\mathbb{C} \times \dots \times \mathbb{C}}_{|\Sigma|}$, such that

the diagram

$$\begin{array}{ccc} R & \xrightarrow{\delta} & \mathbb{C}^\Sigma \\ \text{Id} \downarrow & & \downarrow F_\infty \\ R & \xrightarrow{\delta} & \mathbb{C}^\Sigma \end{array}$$

commutes (here by δ we mean the natural map to the product induced by the family of maps Σ).

Let X be an arithmetic variety over R , i.e. a scheme of finite type over R , which is flat, quasi-projective and regular. As usual we write

$$X(\mathbb{C}) := \coprod_{\sigma \in \Sigma} (X \times_{R, \sigma} \mathbb{C})(\mathbb{C}).$$

We notice that F_∞ induces an involution $F_\infty : X(\mathbb{C}) \rightarrow X(\mathbb{C})$. Let $p, q \in \mathbb{N}$ and y be any cycle in $Z^p(X)$ with $Y := \text{supp } y$. We denote by:

- $D^{p,p}(X_{\mathbb{R}})$ the \mathbb{R} -vector space of real currents ζ on $X(\mathbb{C})$ of type (p, p) such that $F_\infty^* \zeta = (-1)^p \zeta$,
- $\tilde{D}^{p,p}(X_{\mathbb{R}})$ the quotient $D^{p,p}(X_{\mathbb{R}})/(\text{Im } \partial + \text{Im } \bar{\partial})$,
- $E^{p,p}(X_{\mathbb{R}})$ the \mathbb{R} -vector space of smooth real forms ω on $X(\mathbb{C})$ of type (p, p) such that $F_\infty^* \omega = (-1)^p \omega$,
- $\tilde{E}^{p,p}(X_{\mathbb{R}})$ the quotient $E^{p,p}(X_{\mathbb{R}})/(\text{Im } \partial + \text{Im } \bar{\partial})$,
- $\text{CH}^q(X)$ the q -th ordinary Chow group of X ,
- $\widehat{\text{CH}}^q(X)$ the q -th arithmetic Chow group of X .

We also fix the following notations for the analytic real Deligne cohomology and the Deligne-Beilinson cohomology (see [EV] for the definitions):

- $H_{D, \text{an}}^q(X(\mathbb{C}), \mathbb{R}(p))$ the q -th analytic real Deligne cohomology \mathbb{R} -vector space,

- $H_{D,\text{an}}^q(X_{\mathbb{R}}, \mathbb{R}(p))$ the set $\{\gamma \in H_{D,\text{an}}^q(X(\mathbb{C}), \mathbb{R}(p)) | F_{\infty}^* \gamma = (-1)^p \gamma\}$,
- $H_D^q(X(\mathbb{C}), \mathbb{R}(p))$ the q -th Deligne-Beilinson cohomology \mathbb{R} -vector space,
- $H_{D,Y}^q(X(\mathbb{C}), \mathbb{R}(p))$ the q -th Deligne-Beilinson cohomology \mathbb{R} -vector space with support in Y ,
- $H_D^q(X_{\mathbb{R}}, \mathbb{R}(p))$ the set $\{\gamma \in H_D^q(X(\mathbb{C}), \mathbb{R}(p)) | F_{\infty}^* \gamma = (-1)^p \gamma\}$,
- $H_{D,Y}^q(X_{\mathbb{R}}, \mathbb{R}(p))$ the set $\{\gamma \in H_{D,Y}^q(X(\mathbb{C}), \mathbb{R}(p)) | F_{\infty}^* \gamma = (-1)^p \gamma\}$.

3. BURGOS' ARITHMETIC CHOW GROUPS

For the convenience of the reader we recall in this section some definitions and basic facts of Burgos' arithmetic Chow groups (cf. [Bur97] for more details).

3.1. Smooth forms with logarithmic singularities along infinity. Let us start with the definition of smooth differential forms with logarithmic singularities along infinity.

Let V be a smooth algebraic variety over \mathbb{C} and let D be a divisor with normal crossings on V . Let us write $W = V \setminus D$ and let $j : W \hookrightarrow V$ be the inclusion. Let \mathcal{E}_V^* be the sheaf of complex C^∞ differential forms on V and let $E^*(V)$ denote $\Gamma(V, \mathcal{E}_V^*)$. The complex of sheaves $\mathcal{E}_V^*(\log D)$ is the sub- \mathcal{E}_V^* algebra of $j_* \mathcal{E}_W^*$ generated locally by the sections

$$\log z_i \bar{z}_i, \quad \frac{dz_i}{z_i}, \quad \frac{d\bar{z}_i}{\bar{z}_i}, \quad \text{for } i = 1, \dots, M,$$

where $z_1 \cdots z_M = 0$ is a local equation of D .

Let us write $E_V^*(\log D) = \Gamma(V, \mathcal{E}_V^*(\log D))$ and let $E_{V,\mathbb{R}}^*(\log D)$ be the subcomplex of real forms.

Let I be the category of all smooth compactifications of V . That is, an element $(\tilde{V}_\alpha, i_\alpha)$ of I is a smooth complex variety \tilde{V}_α and an immersion $i_\alpha : V \hookrightarrow \tilde{V}_\alpha$ such that $D_\alpha = \tilde{V}_\alpha - i_\alpha(V)$ is a normal crossing divisor. The morphisms of I are the maps $f : \tilde{V}_\alpha \rightarrow \tilde{V}_\beta$ such that $f \circ i_\alpha = i_\beta$. The opposed category I^o is directed (see [Del71]). The complex of smooth differential forms with logarithmic singularities along infinity $E_{\log}^*(V)$ is defined as

$$E_{\log}^*(V) = \varinjlim_{\alpha \in I^o} E_{\tilde{V}_\alpha}^*(\log D_\alpha)$$

This complex is a subcomplex of $E^*(V)$ and we shall denote by $E_{\log,\mathbb{R}}^*(V)$ the corresponding real subcomplex.

The complex $E_{\log}^*(V)$ has a natural bigrading

$$E_{\log}^*(V) = \bigoplus E_{\log}^{p,q}(V).$$

The Hodge filtration of this complex is defined by

$$F^p E_{\log}^n(V) = \bigoplus_{\substack{p' \geq p \\ p' + q' = n}} E_{\log}^{p',q'}(V).$$

We write $E_{\log, \mathbb{R}}^*(V, p) = (2\pi i)^p E_{\log, \mathbb{R}}^*(V) \subseteq E_{\log}^*(V)$.

An important property of the complex $E_{\log}^*(V)$ is that it is strictly related to Deligne-Beilinson cohomology. More precisely, let us consider the following complex

$$E_{\log, \mathbb{R}}^*(V, p)_D := s(u : E_{\log, \mathbb{R}}^*(V, p) \oplus F^p E_{\log}^*(V) \rightarrow E_{\log}^*(V))$$

where $u(a, b) = b - a$ and $s(u)$ stands for the simple complex of the map u . Then

$$H_D^*(V, \mathbb{R}(p)) = H^*(E_{\log, \mathbb{R}}^*(V, p)_D).$$

3.2. Definition of Burgos' arithmetic Chow groups. Now fix $p \in \mathbb{N}^*$. Any cycle $y \in Z^p(X)$ defines a class in

$$\rho(y) \in H_{D,Y}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p))$$

where $Y = \text{supp } y$. Any $g \in E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}), p-1)$ with $\partial \bar{\partial} g \in E^{2p}(X(\mathbb{C}))$, also defines a class

$$\{-2\partial \bar{\partial} g, g\} \in H_{D,Y}^{2p}(X, \mathbb{R}(p)).$$

The space of Green forms associated with y is then

$$\text{GE}_y^p(X_{\mathbb{R}}) := \left\{ g \in E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}), p-1) \left| \begin{array}{l} -2\partial \bar{\partial} g \in E^{2p}(X(\mathbb{C})) \\ \{-2\partial \bar{\partial} g, g\} = \rho(y) \\ F_{\infty}^* g = \bar{g} \end{array} \right. \right\} / (\text{Im } \partial + \text{Im } \bar{\partial}).$$

The group of arithmetic cycles in the sense of Burgos is

$$\hat{Z}^p(X) := \{(y, \tilde{g}) \mid y \in Z^p(X) \text{ and } \tilde{g} \in \text{GE}_y^p(X_{\mathbb{R}})\}.$$

If W is a codimension $p-1$ irreducible subvariety of X and $f \in k(W)^*$, we have a well-defined subvariety $W(\mathbb{C})$ of $X(\mathbb{C})$ and a well-defined function $f_{\mathbb{C}} \in k(W(\mathbb{C}))^*$. To $f_{\mathbb{C}}$ is associated a class

$$\rho(f_{\mathbb{C}}) \in H_D^{2p-1}((X \setminus F)_{\mathbb{R}}, \mathbb{R}(p))$$

where $F = \text{supp } \text{div } f$. Since $H_D^{2p-1}((X \setminus F)_{\mathbb{R}}, \mathbb{R}(p))$ is the same as

$$\left\{ g \in E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus F(\mathbb{C}), p-1) \mid \partial \bar{\partial} g = 0 \text{ and } F_{\infty}^* g = \bar{g} \right\} / (\text{Im } \partial + \text{Im } \bar{\partial}),$$

then one can check that $\rho(f_{\mathbb{C}})$ defines an element in $\text{GE}_{\text{div } f}^p(X_{\mathbb{R}})$, which we denote by $\text{b}(\rho(f_{\mathbb{C}}))$. Let $\widehat{\text{Rat}}^p(X)$ be the subgroup of $\hat{Z}^p(X)$ generated by the elements of the form

$$\widehat{\text{div}} f = (\text{div } f, -\text{b}(\rho(f_{\mathbb{C}}))).$$

The arithmetic Chow group $\widehat{\text{CH}}_{\log}^p(X)$ in the sense of Burgos is

$$\widehat{\text{CH}}_{\log}^p(X) := \hat{Z}^p(X) / \widehat{\text{Rat}}^p(X).$$

The class of an element $(y, \tilde{g}) \in \hat{Z}^p(X)$ will be denoted by $[y, \tilde{g}]$.

3.3. Two important properties. Burgos' arithmetic Chow groups fit in the following exact sequence

$$(3) \quad \text{CH}^{p, p-1}(X) \xrightarrow{\rho} H_D^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{a} \widehat{\text{CH}}_{\log}^p(X) \xrightarrow{(\zeta, -\omega)} \text{CH}^p(X) \oplus ZE_{\log}^{p, p}(X_{\mathbb{R}})$$

where $\text{CH}^{p, p-1}$ is Gillet-Soulé's version of one of Bloch's higher Chow groups, ρ is defined in the proof of Corollary 6.3 in [Bur97], the map a sends the class of \tilde{f} to $[0, \tilde{f}]$ and $(\zeta, -\omega)([y, \tilde{g}]) = (y, 2\partial\bar{\partial}g)$. Later we will make use of the fact

$$\text{CH}^{p, p-1}(X)_{\mathbb{Q}} \cong H_{\mathcal{M}}^{2p-1}(X, p)$$

(see section 1.4 in [BGKK07] for this).

Furthermore there exists a homomorphism

$$\psi_X : \widehat{\text{CH}}_{\log}^p(X) \rightarrow \widehat{\text{CH}}^p(X)$$

which is compatible with pull-backs and is an isomorphism if X is proper over R .

In particular for any $y \in Z^p(X)$, we have a commutative diagram

$$\begin{array}{ccc} \widehat{\text{CH}}_{\log}^p(X) & \xrightarrow{i^*} & \widehat{\text{CH}}_{\log}^p(X \setminus Y) \\ \psi_X \downarrow & & \downarrow \psi_{X \setminus Y} \\ \widehat{\text{CH}}^p(X) & \xrightarrow{i^*} & \widehat{\text{CH}}^p(X \setminus Y) \end{array}$$

where $Y = \text{supp } y$ and the map i is the immersion $X \setminus Y \hookrightarrow X$.

4. AN INTERMEDIATE RESULT

Theorem 3. *Let X/R be a proper arithmetic variety, y any cycle in $Z^p(X)$ and h any Green current for y . Then there exists a representative h_0 of h belonging to $E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}))$, where $Y = \text{supp } y$. If $\omega([y, h]) := \delta_y + \text{dd}^c h = 0$, then $\frac{h_0}{2(2\pi i)^{1-p}}$ defines a class*

$$\frac{\widetilde{h_0}}{2(2\pi i)^{1-p}} \in H_D^{2p-1}((X \setminus Y)_{\mathbb{R}}, \mathbb{R}(p))$$

which does not depend on the choice of h_0 and verifies

$$a \left(\frac{\widetilde{h_0}}{2(2\pi i)^{1-p}} \right) = i^* (\psi_X^{-1}([y, h])).$$

Proof. If we denote by $\text{GC}_X(y)$ the space of Green currents for the cycle y , Theorem 5.9 in [Bur97] tells us that there is a natural isomorphism

$$\text{GE}_y^p(X_{\mathbb{R}}) \rightarrow \text{GC}_X(y)$$

which sends \tilde{g} to the class of the current $2(2\pi i)^{1-p}[g]$, where $[g]$ sends a form ω to

$$[g](\omega) = \int_{X(\mathbb{C})} \omega \wedge g.$$

Let $\widetilde{h_{\log}} \in \text{GE}_y^p(X_{\mathbb{R}})$ be the inverse image of h by this isomorphism. Any element in $2(2\pi i)^{1-p}\widetilde{h_{\log}}$ gives a representative of h belonging to $E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}))$.

If $\delta_y + \text{dd}^c h = 0$, we deduce

$$0 = (\delta_y + \text{dd}^c h)|_{X(\mathbb{C}) \setminus Y(\mathbb{C})} = (\text{dd}^c h)|_{X(\mathbb{C}) \setminus Y(\mathbb{C})} = 2(2\pi i)^{1-p} \text{dd}^c (\widetilde{h_{\log}}).$$

Therefore we have a well-defined element $\widetilde{h_{\log}}$ in

$$\left\{ g \in E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}), p-1) \left| \begin{array}{l} \partial \bar{\partial} g = 0 \\ F_{\infty}^* g = \bar{g} \end{array} \right. \right\} / (\text{Im } \partial + \text{Im } \bar{\partial})$$

i.e. in $H_D^{2p-1}((X \setminus Y)_{\mathbb{R}}, \mathbb{R}(p))$.

By definition of $\widetilde{h_{\log}}$ and by definition of the pullback in Burgos' theory, we have

$$i^* (\psi_X^{-1}([y, h])) = i^* ([y, \widetilde{h_{\log}}]) = [0, \widetilde{h_{\log}}] = a (\widetilde{h_{\log}}).$$

To conclude the proof, take any representative h_0 of h in $E_{\log, \mathbb{R}}^{p-1, p-1}(X(\mathbb{C}) \setminus Y(\mathbb{C}))$. We want to show that $\frac{h_0}{2(2\pi i)^{1-p}}$ defines a class $\frac{\widetilde{h_0}}{2(2\pi i)^{1-p}}$ in $H_D^{2p-1}((X \setminus Y)_{\mathbb{R}}, \mathbb{R}(p))$ and that this class is $\widetilde{h_{\log}}$. Since $\delta_y + \text{dd}^c h_0 = 0$, Proposition 6.5 in [BGKK07] implies

that $\frac{\widetilde{h}_0}{2(2\pi i)^{1-p}}$ is an element in $\mathrm{GE}_Y^p(X_{\mathbb{R}})$ (please notice that the definitions of current associated to a cycle and of current associated to a differential form used in Burgos' paper slightly differ from the ones used here). This element must coincide with $\widetilde{h_{\log}}$, since both have image h in $\mathrm{GC}_X(y)$. \square

Remark 4. *Tensoring with \mathbb{Q} over \mathbb{Z} is an exact operation, so the exact sequence (3) shows that*

$$\begin{aligned} i^* (\psi_X^{-1}([y, h])) = 0 \in \widehat{\mathrm{CH}}_{\log}^p(X \setminus Y)_{\mathbb{Q}} &\Leftrightarrow \frac{\widetilde{h}_0}{2(2\pi i)^{1-p}} \in \rho(\mathrm{CH}^{p,p-1}(X \setminus Y)_{\mathbb{Q}}) \\ &\Leftrightarrow \frac{\widetilde{h}_0}{2(2\pi i)^{1-p}} \in \mathrm{cyc}\left(H_{\mathcal{M}}^{2p-1}(X \setminus Y, p)\right) \end{aligned}$$

5. THE CASE OF AN ABELIAN SCHEME OVER AN ARITHMETIC VARIETY

In this section we explain in detail how the class

$$([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}$$

actually gives rise to an element in $H_{D,\mathrm{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$.

Let S be an arithmetic variety over R and let $\pi : \mathcal{A} \rightarrow S$ be an abelian scheme over S of relative dimension g . We shall write as usual $\mathcal{A}^\vee \rightarrow S$ for the dual abelian scheme. We let $\varepsilon_{\mathcal{A}} = \varepsilon$ be the zero-section of $\pi : \mathcal{A} \rightarrow S$. We shall denote by $S_0 = S_{0,\mathcal{A}} = \varepsilon_{\mathcal{A}}(S)$ the reduced closed subscheme of \mathcal{A} , which is the image of $\varepsilon_{\mathcal{A}}$. We write \mathcal{P} for the Poincaré bundle on $\mathcal{A} \times_S \mathcal{A}^\vee$ and $p_1 : \mathcal{A} \times_S \mathcal{A}^\vee \rightarrow \mathcal{A}$ for the first projection. Since \mathcal{A} and S are arithmetic varieties over R , we have two well-defined complex manifolds $\mathcal{A}(\mathbb{C})$ and $S(\mathbb{C})$ and two well-defined \mathbb{R} -vector spaces $D^{g-1,g-1}(\mathcal{A}_{\mathbb{R}})$ and $E^{g-1,g-1}(\mathcal{A}_{\mathbb{R}})$. We endow the Poincaré bundle \mathcal{P} with the unique metric $h_{\mathcal{P}}$ such that the canonical rigidification of \mathcal{P} along the zero-section $\varepsilon \times \mathrm{Id}_{\mathcal{A}^\vee} : \mathcal{A}^\vee \rightarrow \mathcal{A} \times_S \mathcal{A}^\vee$ is an isometry and such that the curvature form of $h_{\mathcal{P}}$ is translation invariant along the fibres of the map $\mathcal{A}(\mathbb{C}) \times_{S(\mathbb{C})} \mathcal{A}^\vee(\mathbb{C}) \rightarrow \mathcal{A}^\vee(\mathbb{C})$. We write $\overline{\mathcal{P}} = (\mathcal{P}, h_{\mathcal{P}})$ for the resulting hermitian line bundle.

Theorem 5. *There is a unique class of currents $\mathfrak{g}_{\mathcal{A}^\vee} \in \tilde{D}^{g-1,g-1}(\mathcal{A}_{\mathbb{R}})$ with the following three properties:*

- (1) *Any element of $\mathfrak{g}_{\mathcal{A}^\vee}$ is a Green current for $S_0(\mathbb{C})$.*
- (2) *The identity $(S_0, \mathfrak{g}_{\mathcal{A}^\vee}) = (-1)^g p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}$ holds in $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$.*

(3) *The identity $\mathfrak{g}_{\mathcal{A}^\vee} = [n]_* \mathfrak{g}_{\mathcal{A}^\vee}$ holds for all $n \geq 2$.*

Here $[n] : \mathcal{A} \rightarrow \mathcal{A}$ is the multiplication-by- n morphism and $\widehat{\text{ch}}(\overline{\mathcal{P}})$ is the arithmetic Chern character of the hermitian bundle $\overline{\mathcal{P}}$.

Let us fix N a positive natural number and call

$$T := [N]^*(S_0, \mathfrak{g}_{\mathcal{A}^\vee}) - N^{2g}(S_0, \mathfrak{g}_{\mathcal{A}^\vee}).$$

We will write $[T]$ for its class in $\widehat{\text{CH}}^g(\mathcal{A})$. We denote by $\mathcal{A}[N]$ the N -torsion of \mathcal{A} , by i the immersion $\mathcal{U} := \mathcal{A} \setminus \mathcal{A}[N] \hookrightarrow \mathcal{A}$ and by Γ the restriction of the class of currents

$$[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$$

to $\mathcal{U}(\mathbb{C})$. Then the morphism

$$i^* : \widehat{\text{CH}}^g(\mathcal{A}) \rightarrow \widehat{\text{CH}}^g(\mathcal{U})$$

sends $[T]$ to $[0, \Gamma]$. We recall the fundamental exact sequence

$$(4) \quad \text{CH}^{g, g-1}(\mathcal{U}) \xrightarrow{\rho_{\text{an}}} \tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}}) \xrightarrow{a} \widehat{\text{CH}}^g(\mathcal{U}) \longrightarrow \text{CH}^g(\mathcal{U}) \longrightarrow 0$$

(See Theorem 3.3.5 in [GS90]) for this). Here the map a sends the class of ω to $[0, \omega]$.

By construction, ρ_{an} is the following composite function

$$\text{CH}^{g, g-1}(\mathcal{U}) \xrightarrow{\rho} H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{\text{forgetful}} H_{D, \text{an}}^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \longrightarrow \tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$$

where the third map is a natural inclusion. Indeed we have

$$H_{D, \text{an}}^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) = \{c \in (2\pi i)^{g-1} E^{g-1, g-1}(\mathcal{U}_{\mathbb{R}}) \mid \partial \bar{\partial} g = 0\} / (\text{Im} \partial + \text{Im} \bar{\partial})$$

and the class of c is sent to the class of $c/(2\pi i)^{g-1}$.

Since the image of $[0, \Gamma]$ in $\text{CH}^g(\mathcal{U})$ is 0, there exists an element in $\tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$ sent to $[0, \Gamma]$ by a . We know how to construct such an element: thanks to Theorem 1.3.5 and Theorem 1.2.4 in [GS90], the class of currents $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ has a representative γ in $E^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$. Now if γ' is any another representative smooth on $\mathcal{U}(\mathbb{C})$, we have that $\gamma - \gamma'$ is a smooth form on $\mathcal{U}(\mathbb{C})$ and $\gamma - \gamma' = \partial c_1 + \bar{\partial} c_2$ for some currents c_1 and c_2 . Therefore there exist two smooth forms ω_1 and ω_2 such that $\gamma - \gamma' = \partial \omega_1 + \bar{\partial} \omega_2$ (see Theorem 1.2.2 (ii) in [GS90]). This implies that the class $\tilde{\gamma} \in \tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$ does not depend on γ . We have

$$a(\tilde{\gamma}) = [0, \tilde{\gamma}] = [0, \Gamma].$$

The calculations we do to prove the next proposition are basically the same the reader can find in Lemma 2.4.5 in [KR14].

Proposition 6. *The element $[T]$ is zero in $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$.*

Proof. By Theorem 5, we have

$$[T] = (-1)^g([N]^* - N^{2g}) \left(p_{1,*} \left(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}) \right)^{(g)} \right)$$

in $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$. Now consider the following square of schemes over S :

$$\begin{array}{ccc} \mathcal{A} \times_S \mathcal{A}^\vee & \xrightarrow{N \times \mathrm{Id}} & \mathcal{A} \times_S \mathcal{A}^\vee \\ p_1 \downarrow & & \downarrow p_1 \\ \mathcal{A} & \xrightarrow{N} & \mathcal{A} \end{array}$$

For any Q scheme over S and any pair of morphisms of schemes over S

$$\begin{aligned} (\zeta, \sigma) : Q &\rightarrow \mathcal{A} \times_S \mathcal{A}^\vee \\ \eta : Q &\rightarrow \mathcal{A} \end{aligned}$$

we can define $(\eta, \sigma) : Q \rightarrow \mathcal{A} \times_S \mathcal{A}^\vee$. It is easy to see that this is a morphism of schemes over S and that, if $N \circ \eta = \zeta$, it verifies $(N \times \mathrm{Id}) \circ (\eta, \sigma) = (\zeta, \sigma)$ and $p_1 \circ (\eta, \sigma) = \eta$. Furthermore, (η, σ) is the unique morphism of schemes over S with these properties. Therefore the square above is cartesian. Since the direct image in arithmetic Chow theory is naturally compatible with smooth base change, we have

$$\begin{aligned} (-1)^g [N]^* \left(p_{1,*} (\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)} \right) &= (-1)^g p_{1,*} \left[\left((N \times \mathrm{Id})^* \widehat{\mathrm{ch}}(\overline{\mathcal{P}}) \right)^{(2g)} \right] \\ &= (-1)^g p_{1,*} \left[\left(\widehat{\mathrm{ch}}((N \times \mathrm{Id})^*(\overline{\mathcal{P}})) \right)^{(2g)} \right]. \end{aligned}$$

From the definition of dual abelian scheme we know that there is an isomorphism between the group $\mathrm{End}(\mathcal{A}, \mathcal{A})$ and the group of isomorphism classes of invertible sheaves \mathcal{L} on $\mathcal{A} \times_S \mathcal{A}^\vee$ with rigidification along $\varepsilon_{\mathcal{A}^\vee}$ such that $\mathcal{L} \otimes k(a)$ is algebraically equivalent to 0 in A_a for all $a \in \mathcal{A}^\vee$. Via this isomorphism, a map $\phi : \mathcal{A} \rightarrow \mathcal{A}$ is sent

to $(\phi \times id_{\mathcal{A}^\vee})^*(\mathcal{P})$, so the image of $\text{Id}_{\mathcal{A}}$ is \mathcal{P} . Then the image of $[N]$, i.e $(N \times \text{Id})^*(\mathcal{P})$, must coincide with $\mathcal{P}^{\otimes N}$. Therefore:

$$(-1)^g [N]^* \left(p_{1,*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)} \right) = (-1)^g p_{1,*} \left[\left(\widehat{\text{ch}}(\overline{\mathcal{P}}^{\otimes N}) \right)^{(2g)} \right]$$

and

$$\begin{aligned} [T] &= (-1)^g \left[p_{1,*} \left[\left(\widehat{\text{ch}}(\overline{\mathcal{P}}^{\otimes N}) \right)^{(2g)} \right] - N^{2g} p_{1,*} \left(\widehat{\text{ch}}(\overline{\mathcal{P}}) \right)^{(g)} \right] \\ &= (-1)^g \left[p_{1,*} \left(\frac{\widehat{c}_1(\overline{\mathcal{P}}^{\otimes N})^{2g}}{2g!} \right) - N^{2g} p_{1,*} \left(\frac{\widehat{c}_1(\overline{\mathcal{P}})^{2g}}{2g!} \right) \right] \\ &= (-1)^g \left[p_{1,*} \left(\frac{N^{2g} \widehat{c}_1(\overline{\mathcal{P}})^{2g}}{2g!} \right) - N^{2g} p_{1,*} \left(\frac{\widehat{c}_1(\overline{\mathcal{P}})^{2g}}{2g!} \right) \right] \\ &= 0 \end{aligned}$$

where $\widehat{c}_1(\cdot)$ refers to the first arithmetic Chern class of a hermitian bundle. Notice that we used the multiplicativity of $\widehat{c}_1(\cdot)$. \square

Corollary 7. *The class $(2\pi i)^{g-1} \tilde{\gamma} \in (2\pi i)^{g-1} \tilde{E}^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$ is in the image of*

$$\text{cyc}_{\text{an}} : H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g) \rightarrow H_{D, \text{an}}^{2g-1}(\mathcal{U}, \mathbb{R}(g)).$$

Proof. Proposition 6 implies that

$$0 = i^*([T]) = [0, \Gamma] = a(\tilde{\gamma})$$

in $\widehat{\text{CH}}^g(\mathcal{U})_{\mathbb{Q}}$ and the exactness of sequence (4) gives us $\tilde{\gamma} \in \rho_{\text{an}}(\text{CH}^{g, g-1}(\mathcal{U})_{\mathbb{Q}})$. By the definition of ρ_{an} , we obtain that

$$(2\pi i)^{g-1} \tilde{\gamma} \in (\text{forgetful} \circ \rho)(\text{CH}^{g, g-1}(\mathcal{U})_{\mathbb{Q}}).$$

This is exactly what we wanted to prove, once one identifies $H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)$ with $\text{CH}^{g, g-1}(\mathcal{U})_{\mathbb{Q}}$. \square

6. THE MAIN RESULT

We are now able to apply Theorem 3 to our situation. We assume that S is proper over R , so \mathcal{A} is proper over R .

Corollary 8. *There exists a representative of $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ belonging to $E_{\log, \mathbb{R}}^{g-1, g-1}(\mathcal{U}(\mathbb{C}))$. Any such η defines an element*

$$\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g) \rightarrow H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \right)$$

which does not depend on the choice of η and verifies

$$a \left(\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \right) = i^* (\psi_{\mathcal{A}}^{-1}([y, h])) = 0.$$

Furthermore

$$\text{forgetful} \left(\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \right) = \frac{\tilde{\gamma}}{2(2\pi i)^{1-g}}.$$

Proof. To prove the first assertion we apply Theorem 3 and Remark 4 with X equal to \mathcal{A} and (y, h) equal to T . The hypotheses are satisfied since $[T] = 0$ in $\widehat{\text{CH}}^g(\mathcal{A})_{\mathbb{Q}}$, so $i^* (\psi_{\mathcal{A}}^{-1}([T])) = 0$ in $\widehat{\text{CH}}_{\log}^g(\mathcal{U})_{\mathbb{Q}}$.

To prove the second assertion, it is enough to notice that any representative η of $[N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \mathfrak{g}_{\mathcal{A}^\vee}$ belonging to $E_{\log, \mathbb{R}}^{g-1, g-1}(\mathcal{U}(\mathbb{C}))$, also belongs to $E^{g-1, g-1}(\mathcal{U}_{\mathbb{R}})$. \square

Remark 9. *A natural analog of the operator $\text{tr}_{[a]}$ operates on Deligne-Beilinson cohomology and the map cyc intertwines this operator with $\text{tr}_{[a]}$. Therefore from the existence of the Jordan decomposition and the fact*

$$\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g) \rightarrow H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \right)$$

we deduce

$$\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \in \text{Im} \left(\text{cyc} : H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)^{(0)} \rightarrow H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g)) \right).$$

We are ready to prove our main result.

Theorem 10. *Let $\text{pol}^0 \in H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)$ be the zero step of the motivic polylogarithm on \mathcal{A} (as defined in the Introduction). Then*

$$-2 \cdot \text{cyc}(\text{pol}^0) = \frac{\tilde{\eta}}{(2\pi i)^{1-g}}$$

in $H_D^{2g-1}(\mathcal{U}_{\mathbb{R}}, \mathbb{R}(g))$.

Proof. We start noticing that

$$\text{cyc}_{\text{an}}(\text{pol}^0) = -\frac{\tilde{\gamma}}{2(2\pi i)^{1-g}} = \text{forgetful} \left(-\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} \right).$$

By Remark 9 we know that

$$-\frac{\tilde{\eta}}{2(2\pi i)^{1-g}} = \text{cyc}(l)$$

for some $l \in H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)^{(0)}$. Therefore we have

$$\text{cyc}_{\text{an}}(\text{pol}^0) = \text{cyc}_{\text{an}}(l)$$

and the injectivity of cyc_{an} on $H_{\mathcal{M}}^{2g-1}(\mathcal{U}, g)^{(0)}$ implies that $\text{pol}^0 = l$. We then obtain

$$\text{cyc}(\text{pol}^0) = \text{cyc}(l) = -\frac{\tilde{\eta}}{2(2\pi i)^{1-g}}.$$

□

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